GENERALIZATION OF L. V. KANTOROVICH'S METHOD APPLIED TO BOUNDARY-VALUE PROBLEMS OF HEAT CONDUCTION

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We give a generalization of Kantorovich's method of reduction to an ordinary differential equation [1], as applied to boundary-value problems of heat conduction.

L. V. Kantorovich's method of reduction to an ordinary differential equation [1] has been applied extensively in analyzing boundary-value problems in mathematical physics thanks to a number of advantages that it possesses over other approximate methods.

Most of the approximate methods, for example, the method of Ritz–Galerkin, the method of least squares, the Trefftz method, the method of collocation, etc., require an a priori knowledge of the form of the solution and permit only in an optimum manner the determination of the undetermined constants in the coefficients appearing in the structure of the given solution.

We propose here a generalization of Kantorovich's method suitable for boundary-value problems of stationary heat conduction, which makes it possible to take into account the properties of the boundary-value problem operator with respect to all the unknown variables and to obtain much higher accuracy, even in the first approximation.

We consider the stationary temperature field inside an anisotropic parallelepiped with uniformly distributed energy sources subject to boundary conditions of the third kind. The temperature field may be described by the equation

$$\lambda_x \frac{\partial^2 t}{\partial x^2} + \lambda_y \frac{\partial^2 t}{\partial y^2} + \lambda_z \frac{\partial^2 t}{\partial z^2} + q = 0 \tag{1}$$

inside the parallelepiped $|x| \leq l_x$, $|y| \leq l_y$, $|z| \leq l_z$, and by the conditions

$$\left[\frac{\partial t}{\partial i} - \frac{\alpha_i}{\lambda_i}(t - t_c)\right]_{i=-l_i} = 0, \ \left[\frac{\partial t}{\partial i} + \frac{\alpha_i}{\lambda_i}(t - t_c)\right]_{i=l_i} = 0 \tag{2}$$

on its boundaries. Here $i = x, y, z$.

We introduce the following dimensionless quantities:

$$j = \frac{i}{l_i}; \ j = \xi, \eta, \zeta; \ \mathrm{Bi} = \frac{\alpha_i L_i}{\lambda_i};$$

$$\varepsilon_i = \frac{\lambda_i}{\lambda_m}\left(\frac{L_m}{L_i}\right)^2; \ \vartheta = t - t_a; \ N = \frac{\vartheta \lambda_m}{q L_m^2}.$$

As the scaling parameters λ_m and L_m, we choose, respectively, the quantities λ_z and L_z. Taking into account the symmetry of the problem and also the notation introduced above, we find it appropriate to consider the following boundary-value problem, equivalent to the initial one:

$$\varepsilon_x \frac{\partial^2 N}{\partial \xi^2} + \varepsilon_y \frac{\partial^2 N}{\partial \eta^2} + \frac{\partial^2 N}{\partial \zeta^2} + 1 = 0, \tag{3}$$

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$$\left[\frac{\partial N}{\partial j} + \mathrm{Bi}\, N\right]_{j=1} = 0, \quad \left[\frac{\partial N}{\partial j}\right]_{j=0} = 0. \tag{4}$$

The essence of the method that we propose, as applied to the boundary-value problem (3)-(4), consists in the following. We average the unknown function $N(\xi, \eta, \zeta)$ with respect to two variables, first with respect to η and ζ, for which purpose we apply to the differential equation (3) and the conditions (4) the averaging operator $I_{\eta\zeta}$:

$$I_{\eta\zeta}[N] = \int_0^1\int_0^1 N d\eta d\zeta = <N_{\eta\zeta}>,$$

$$I_{\eta\zeta}\left[\varepsilon_x \frac{\partial^2 N}{\partial \xi^2}\right] = \varepsilon_x \frac{d^2 <N_{\eta\zeta}>}{d\xi^2},$$

$$I_{\eta\zeta}\left[\varepsilon_y \frac{\partial^2 N}{\partial \eta^2}\right] = \varepsilon_y \int_0^1 d\zeta \int_0^1 \frac{\partial^2 N}{\partial \eta^2} d\eta = \varepsilon_y \int_0^1 \left[\frac{\partial N}{\partial \eta}\right]_{\eta=0}^{\eta=1} d\zeta.$$

In accord with the boundary conditions (4),

$$\left[\frac{\partial N}{\partial \eta}\right]_{\eta=0}^{\eta=1} = -B_y N(\eta = 1),$$

consequently,

$$I_{\eta\zeta}\left[\varepsilon_y \frac{\partial^2 N}{\partial \eta^2}\right] = -\varepsilon_y B_y \int_0^1 N(\eta = 1)\, d\zeta = \varepsilon_y B_y \frac{\int_0^1 N(\eta = 1)\, d\zeta}{<N_{\eta\zeta}>} <N_{\eta\zeta}> = -\varepsilon_y B_y \psi_y <N_{\eta\zeta}>.$$

Here we have introduced the coefficient $\psi_y = \int_0^1 N(\eta = 1) d\zeta / <N_{\eta\zeta}>$, which characterizes the nonuniformity of the temperature field at the section with abscissa ξ. The coefficient ψ_y is a function of ξ, but from physical considerations it follows that this dependence is a weak one; therefore, we shall henceforth use its average value, given by the equation

$$\psi_y \cong \mathrm{const} = \frac{\int_0^1\int_0^1 N(\xi, \eta = 1, \zeta)\, d\xi d\zeta}{\int_0^1 <N_{\eta\zeta}> d\xi} = \frac{<N_{\xi\zeta}(\eta = 1)>}{N_v}.$$

Proceeding in a completely analogous way, we obtain

$$I_{\eta\zeta}\left[\frac{\partial^2 N}{\partial \zeta^2}\right] = -B_z \psi_z <N_{\eta\zeta}>,$$

$$\psi_z = \frac{<N_{\xi\eta}(\zeta = 1)>}{N_v}, \quad I_{\eta\zeta}[1] = 1.$$

Combining the results obtained in applying the operator $I_{\eta\zeta}$ term-wise to the differential equation (4), we obtain an ordinary differential equation in $<N_{\eta\zeta}>$:

$$\frac{d^2 <N_{\eta\zeta}>}{d\xi^2} - p_x^2 <N_{\eta\zeta}> = -\frac{1}{\varepsilon_x},$$
$$p_x^2 = \frac{\varepsilon_y B_y \psi_y + B_z \psi_z}{\varepsilon_x}. \tag{5}$$

An application of the operator $I_{\eta\zeta}$ to the boundary conditions (4) for $\xi = 0.1$ yields the following conditions for $<N_{\eta\zeta}>$ at the boundary:

$$\left[\frac{d <N_{\eta\zeta}>}{d\xi} + B_x <N_{\eta\zeta}>\right]_{=1} = 0, \quad \left[\frac{d <N_{\eta\zeta}>}{d\xi}\right]_{\xi=0} = 0. \tag{6}$$

Upon integrating Eq. (5) and satisfying the boundary conditions (6), we obtain

$$<N_{\eta\zeta}> = \frac{1}{\varepsilon_x p_x^2} \varphi_\xi, \tag{7}$$

$$\varphi_\xi = 1 - \Omega_x \frac{\operatorname{ch} p_x \xi}{\operatorname{ch} p_x}, \quad \Omega_x = \left(1 + \frac{p_x \operatorname{th} p_x}{B_x}\right)^{-1}.$$

Thus we now know how the solution depends on the variable ξ. Further, using the ideas in Kantorovich's method [1], we seek an approximate solution in the form

$$\tilde{N} = M \langle N_{\eta\zeta} \rangle. \tag{8}$$

To solve the boundary-value problem (3)-(4) is equivalent to minimizing the following functional [1]:

$$J(N) = \int_0^1\int_0^1\int_0^1 \left\{\varepsilon_x \left(\frac{\partial N}{\partial \xi}\right)^2 + \varepsilon_y \left(\frac{\partial N}{\partial \eta}\right)^2 + \left(\frac{\partial N}{\partial \zeta}\right)^2 - 2N\right\} d\xi d\eta d\zeta. \tag{9}$$

Substituting $\tilde{N}$ in the form (8) into the integral and integrating with respect to ξ, we obtain the problem of minimizing a double integral:

$$J(M) = \int_0^1 d\eta \int_0^1 \left\{a_1 \left[\varepsilon_y \left(\frac{\partial M}{\partial \eta}\right)^2 + \left(\frac{\partial M}{\partial \zeta}\right)^2\right] + a_2 M^2 - 2a_3 M \right\} d\zeta,$$

$$a_1 = \frac{1}{\varepsilon_x p_x^2}\left[1 - 2\Omega_x \frac{\operatorname{th} p_x}{p_x} + \frac{\Omega_x}{2}\left(\frac{\operatorname{th} p_x}{p_x} + \frac{1}{\operatorname{ch}^2 p_x}\right)\right], \tag{10}$$

$$a_2 = \frac{1}{2}\left(\frac{\operatorname{th} p_x}{p_x} - \frac{1}{\operatorname{ch}^2 p_x}\right)\Omega_x^2, \quad a_3 = 1 - \Omega_x \frac{\operatorname{th} p_x}{p_x}.$$

The function M, which makes the functional (10) a minimum, is a solution of the boundary-value problem for the equation

$$\varepsilon_y \frac{\partial^2 M}{\partial \eta^2} + \frac{\partial^2 M}{\partial \zeta^2} - \sigma_1 M = -\sigma_2 \tag{11}$$

inside the square $0 \leq \eta \leq 1$, $0 \leq \zeta \leq 1$ and the boundary conditions

$$\left[\frac{\partial M}{\partial j} + \mathrm{Bi}\, M\right]_{j=1} = 0, \quad \left[\frac{\partial M}{\partial j}\right]_{j=0} = 0 \tag{12}$$

on the contour.

Here $\sigma_1 = a_2/a_1$, $\sigma_2 = a_3/a_1$.

Consequently, the initial boundary-value problem (3)-(4) has been reduced to the two-dimensional problem (11)-(12); we solve the latter by applying the averaging scheme presented above.

Applying the averaging operator I_ζ, $I_\zeta[M] = \int_0^1 M d\zeta = \langle M_\zeta \rangle$, to Eq. (11) and the conditions (12), we obtain an ordinary differential equation in $\langle M_\zeta \rangle$:

$$\frac{d^2 \langle M_\zeta \rangle}{d\eta^2} - p_y^2 \langle M_\zeta \rangle = -\frac{\sigma_2}{\varepsilon_y}, \qquad p_y^2 = \frac{B_z \psi_z + \sigma_1}{\varepsilon_y}. \tag{13}$$

Solving Eq. (13) subject to the boundary conditions

$$\left[\frac{d\langle M_\zeta \rangle}{d\eta} + B_y \langle M_\zeta \rangle\right]_{\eta=1} = 0, \quad \left[\frac{d\langle M_\zeta \rangle}{d\eta}\right]_{\eta=0} = 0,$$

we obtain

$$\langle M_\zeta \rangle = \frac{\sigma_2}{\varepsilon_y p_y^2}\varphi_\eta, \qquad \varphi_\eta = 1 - \Omega_y \frac{\operatorname{ch} p_y \eta}{\operatorname{ch} p_y}, \quad \Omega_y = \left(1 + \frac{p_y \operatorname{th} p_y}{B_y}\right)^{-1}. \tag{14}$$

Thus there remains the problem of finding how N depends on the variable ζ and, by the same token, to obtain a first approximation to the solution.

We now represent M, as was also done in the first step, in the form of a product:

$$\tilde{M} = \langle M_\zeta \rangle Q. \tag{15}$$

Substituting $\tilde{M}$ into the integral (10) and then integrating with respect to the variable η, we reduce the problem to one of minimizing the simple integral

$$J(Q) = \int_0^1 \left\{ b_1 \left(\frac{dQ}{d\zeta} \right)^2 + b_2 Q^2 - 2b_3 Q \right\} d\zeta,$$

$$b_1 = a_1 \frac{\sigma_2^2}{\varepsilon_y p_y^2} \left[1 - 2\Omega_y \frac{\operatorname{th} p_y}{p_y} + \frac{\Omega_y}{2} \left(\frac{\operatorname{th} p_y}{p_y} + \frac{1}{\operatorname{ch}^2 p_y} \right) \right], \tag{16}$$

$$b_2 = \sigma_2^2 \left\{ a_1 \frac{\Omega_y^2}{2} \left(\frac{\operatorname{th} p_y}{p_y} - \frac{1}{\operatorname{ch}^2 p_y} \right) + \left[1 - 2\Omega_y \frac{\operatorname{th} p_y}{p_y} + \frac{\Omega_y^2}{2} \left(\frac{\operatorname{th} p_y}{p_y} + \frac{1}{\operatorname{ch}^2 p_y} \right) \right] \right\}.$$

We write out the Euler equation for the functional (16):

$$\frac{d^2 Q}{d\zeta^2} - p_z^2 Q = -\sigma_3,$$

$$p_z^2 = \frac{b_2}{b_1}, \quad \sigma_3 = \frac{b_3}{b_1}, \quad b_3 = a_3 \sigma_2 \left(1 - \Omega_y \frac{\operatorname{th} p_y}{p_y} \right). \tag{17}$$

Solving Eq. (17) subject to the boundary conditions

$$\left[\frac{dQ}{d\zeta} + B_z Q \right]_{\zeta=1} = 0, \quad \left[\frac{dQ}{d\zeta} \right]_{\zeta=0} = 0,$$

which guarantee satisfaction of the conditions (4) at the boundaries $\zeta = 0$ and 1, we obtain

$$Q = \frac{\sigma_3}{p_z^2} \varphi_\zeta,$$

$$\varphi_\zeta = 1 - \Omega_z \frac{\operatorname{ch} p_z \zeta}{\operatorname{ch} p_z}, \quad \Omega_z = \left(1 + p_z \frac{\operatorname{th} p_z}{B_z} \right)^{-1}. \tag{18}$$

Thus the solution of the boundary-value problem (3)-(4), in accord with the Eqs. (8), (7), (15), (14), and (18), has the form

$$\tilde{N} = \frac{\sigma_2 \sigma_3}{\varepsilon_x p_x^2 \varepsilon_y p_y^2 p_z^2} \varphi_\xi \varphi_\eta \varphi_\zeta,$$

$$\varphi_j = 1 - \Omega_i \frac{\operatorname{ch} p_i j}{\operatorname{ch} p_i}, \quad \Omega_i = \left(1 + \frac{p_i \operatorname{th} p_i}{\mathrm{Bi}} \right)^{-1}. \tag{19}$$

Let us study the errors. We use the exact solution, given in [2], for the problem considered above, subject to boundary conditions of the first kind, and we then compare the approximate solution obtained here with this exact solution. In Eq. (19) we put, throughout, Bi = ∞, which corresponds to writing the approximate solution (19) of the problem for boundary conditions of the first kind:

$$\tilde{N}^{\mathrm{I}} = \left[4 \left(1 - \frac{\operatorname{th} p_x}{p_x} \right) \left(1 - \frac{\operatorname{th} p_y}{p_y} \right) \left(1 - \frac{\operatorname{ch} p_x \xi}{\operatorname{ch} p_x} \right) \left(1 - \frac{\operatorname{ch} p_y \eta}{\operatorname{ch} p_y} \right) \right.$$

$$\left. \times \left(1 - \frac{\operatorname{ch} p_z \zeta}{\operatorname{ch} p_z} \right) \right] \Big/ \left[p_z^2 \left(2 - 3 \frac{\operatorname{th} p_x}{p_x} + \frac{1}{\operatorname{ch}^2 p_x} \right) \left(2 - 3 \frac{\operatorname{th} p_y}{p_y} + \frac{1}{\operatorname{ch}^2 p_y} \right) \right]. \tag{20}$$

Here p_x, p_y, p_z are related through the following system of transcendental equations:

$$\varepsilon_x p_x^2 = \varepsilon_y p_y^2 \frac{\operatorname{th} p_y}{p_y - \operatorname{th} p_y} + p_z^2 \frac{\operatorname{th} p_z}{p_z - \operatorname{th} p_z},$$

$$\varepsilon_y p_y^2 = p_z^2 \frac{\operatorname{th} p_z}{p_z - \operatorname{th} p_z} + \varepsilon_x p_x^2 \frac{\dfrac{\operatorname{th} p_x}{p_x} - \dfrac{1}{\operatorname{ch}^2 p_x}}{2 - 3\dfrac{\operatorname{th} p_x}{p_x} + \dfrac{1}{\operatorname{ch}^2 p_x}}, \tag{21}$$

TABLE 1. Relative Errors of Approximate Solutions $\tilde{N}$ at Center of Parallelepiped

L_z/L_y	0			0,2			0,5			1		
L_z/L_x	GK	K	R–G	GK	K	R–G	GK	K	R–G	OK	GK	R–G
0	0,0	25,0	56,6	–1,6	22,3	53,7	–2,8	9,4	36,8	–3,7	4,0	30,4
0,2	–1,6	27,6	53,7	–0,8	24,4	50,2	–2,9	11,6	35,3	–5,4	4,7	26,8
0,5	–2,8	25,2	36,8	–2,7	23,0	35,3	–5,8	12,8	25,8	–6,7	6,7	22,1
1	–3,7	26,1	30,4	–5,1	24,2	27,8	–6,0	17,2	22,1	–7,6	9,8	15,6

$$p_z^2 = \varepsilon_y p_y^2 \frac{\dfrac{\operatorname{th} p_y}{p_y} - \dfrac{1}{\operatorname{ch}^2 p_y}}{2 - 3\dfrac{\operatorname{th} p_y}{p_y} + \dfrac{1}{\operatorname{ch}^2 p_y}} + \varepsilon_x p_x^2 \frac{\dfrac{\operatorname{th} p_x}{p_x} - \dfrac{1}{\operatorname{ch}^2 p_x}}{2 - 3\dfrac{\operatorname{th} p_x}{p_x} + \dfrac{1}{\operatorname{ch}^2 p_x}}.$$

For the center of the parallelepiped we have calculated the corresponding errors $\delta\tilde{N} = [(\tilde{N} - N)/N]100\%$; these are displayed in Table 1.

The system of equations (21) was solved by the method of simple iteration, good convergence being obtained.

We also present in Table 1 the errors incurred in the solutions obtained, in a first approximation, using the methods of Kantorovich and Ritz–Galerkin.

As is evident from Table 1, our method gives much greater accuracy than the Kantorovich and Ritz–Galerkin methods, these being the best known of the approximate methods. In addition, the solution behaves appropriately in the limiting transition to a plate, i.e., it leads to an accurate solution. Finally, in contrast to the indicated classical methods, in the solution we propose no one coordinate direction is preferred over the others.

More recently, in [3], the problem considered above was solved by Galerkin's method, the coordinate functions being chosen in accord with the method of Kantorovich; the numerical results obtained showed that the method proposed makes it possible to increase the precision of Galerkin's method.

Let us compare the error of the approximate solution obtained by our method with that of the first stage of the scheme presented in [3] for a point at the center of an isotropic cube where the largest errors occur.

From Table 1 we see that when $L_z/L_x = L_z/L_y = 1$ the error of our approximate solution is $\delta\tilde{N} = -7.6\%$, while that for the solution given in [3] is $\delta\tilde{N} = -21.3\%$. In addition, the method employed in [3] is more laborious since each step involves all the operations required in both the Kantorovich method and the Galerkin method.

Thus the modified Kantorovich method, as presented here, is more effective.

NOTATION

t	is the temperature inside the parallelepiped;
t_a	is the ambiant temperature;
ϑ	is the heating of the medium;
N	is the dimensionless overheating;
$\tilde{N}$	is an approximate expression for N;
$\tilde{N}^I$	is the value of $\tilde{N}$ in the case of boundary conditions of the first kind;
$\langle N_{\eta\xi} \rangle$	is the value of N averaged over the variables ξ and η;
N_V	is the volume average of N;
$\langle N_{\xi\zeta}(\eta = 1) \rangle$	is the surface average of N at the edge $\eta = 1$;
λ_i	is the thermal conductivity in the direction of the axis O_i;
α_i	is the heat-transfer coefficients at the edges $i = 0, 1$;
q	is the volume density of the energy source;
Bi	is the Biot number;

L_i is the sides of the parallelepiped;
l_i is the half-sides of the parallelepiped;
GK is the solution by the generalized Kantorovich method;
K is the solution by the Kantorovich method;
R–G is the solution by the Ritz–Galerkin method.

LITERATURE CITED

1. L. V. Kantorovich and V. N. Krylov, Approximate Methods of Higher Analysis, Interscience, New York (1958).
2. G. N. Dul'nev and É. M. Semyashkin, Heat Transfer in Radio-Electronic Apparatus [in Russian], Énergiya, Leningrad (1968).
3. A. I. Kaidanov, Inzh.-Fiz. Zh., 18, No. 2 (1970).